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THE USE OF HIGH-ORDER FORMS IN STABILITY ANALYSIS*

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A method is proposed for determining whether forms of arbitrary high order are positive or negative definite in a region of R^n coinciding with one of the coordinate angles. Using such functions, one can then establish various modifications of well-known results of stability theory. A theorem of Grujic /1/ concerning the exponential stability of large-scale systems, generalized to the case of m-th order estimates, yields new zones of absolute stability for the equations of translational motion of an aircraft. Various results are established pertaining to the monotone stability of systems in which the right-hand side is a polynomial of a special kind.

In many problems of stability theory it suffices to construct a Lyapunov function which is positive or negative definite not in the whole space but only in a subspace, namely, a cone. This is a logical approach, for example, in relation to biological communities, since the trajectories of a system describing the dynamics of such interactions never leave the first orthant. Conditions for quadratric forms to be positive (negative) definite in a specific cone - one of the coordinate angles - were studied in /2/. A criterion for a quadratic form to be positive (negative) definite in a certain region of R^n, similar in a sense to the conditions obtained in /2/, was established in /3/ and /4/. Even before that, a criterion was proposed /5/ for a form of order 3 to be positive (negative) definite in one of the coordinate angles. Also worthy of mention is a method described in /6/ to determine whether forms of arbitrary even order are definite in the whole space.

Relying on the concept of a cone coinciding with a coordinate angle, as well as the results and /5/ and /6/, a method can be devised to investigate whether a form of arbitrary high (including odd) order is definite in an orthant of R^n.

1. Definiteness of a form of arbitrary order m in a cone. A cone in R^n coinciding with a coordinate angle will be denoted as follows /7/: $K\{\alpha_{10},\ldots,\alpha_{n0}\}$, where α_{i0} are elements of a basis $\{\alpha_{i0}\}$ taking values +1 and -1, and

$$\alpha_{i0}=\operatorname{sign}x_i,\quad x_i\neq 0;\quad \alpha_{i0}x_i>0$$

Throughout, $i=1,2,\ldots,n$.

In a cone K of the region $H=\{x\colon 0\leqslant\|x\|=|x_1|+\ldots+|x_n|<\infty\}$ we consider an m-th order form

$$W(x)=\sum_{i_1=1}^{n}\cdots\sum_{i_m=i_{m-1}}^{n}A_{i_1\ldots i_m}x_{i_1}\cdots x_{i_m},\quad A_{i_1\ldots i_m}=\text{const}$$

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We perform two substitutions in succession: first, $y_i = \alpha_{i0} x_i$, where α_{i0} are the basis elements of the cone, and $y_i = u_i^2$. This gives a form of even order:

$$\Omega(u) = \sum_{i_1=1}^{n} \cdots \sum_{i_m=i_{m-1}}^{n} B_{i_1 \ldots i_m} u_{i_1}^2 \cdots u_{i_m}^2, \quad B_{i_1 \ldots i_m} = \text{const}$$

which is positive (negative) definite in the whole of R^n if and only if W has the corresponding property in the cone K. Using a result of /6/, we apply the mapping

$$z_1 = u_1^m, \quad z_2 = u_1^{m-1} u_2, \ldots, z_l = u_n^m$$

which carries $\Omega(u)$ into a quadratic function

$$V(z) = \sum_{k=1}^{l} \sum_{j=1}^{l} v_{kj} z_k z_j, \quad v_{kj} = \text{const}$$

Thus, in order to determine whether the m-th order form W is positive (negative) definite in the cone K, we need only analyse the corresponding property of the quadratic form $V(z)$ in the whole space $Z_l \subseteq R^l$. This in turn may be done by well-known methods, e.g., using Sylvester's criterion. We have thus indicated a quite general way to determine whether forms of arbitrarily high order are positive (negative) definite in a cone. It is important to observe, however, that certain specific problems, such as deriving the necessary and sufficient conditions for a third-order form to be positive (negative) definite in a cone coinciding with one of the coordinate angles, can be attacked from a different angle.

2. *Conditions for a third-order form to be positive (negative) definite in a cone.* In a cone K of the region H, consider the third-order form

$$W(x) = \sum_{k=1}^{n} x_k V_k(x), \quad V_k(x) = \sum_{i,j=1}^{n} a_{kij} x_i x_j \tag{2.1}$$
$$a_{kij} = \text{const} \ (i, j, k = 1, 2, \ldots, n)$$

Here $V_k(x)$, for each k, is a quadratic form with matrix $A^{(k)}$, whose elements are a_{kij}. Let $\sigma = (i_1, \ldots, i_r)$ be a sequence from the set Q_{rn} of all strictly increasing sequences, each consisting of r numbers from the set $N = \{1, 2, \ldots, n\}$, and let $A^{(k)}_{[\sigma/\sigma]}$ denote the principal submatrix of the matrix of $V_k(x)$ consisting of the elements of $A^{(k)}$ with incides in σ.

Theorem 1. The form $W(x)$ is positive (negative) definite in the cone K if and only if the system of non-linear algebraic equations

$$\begin{gathered}(A^{(i_k)}_{[\sigma/\sigma]} x_\sigma, x_\sigma) = \alpha_{i_k 0} \lambda, \ i_k \in \sigma, \sigma \in Q_{rn} \\ k = 1, 2, \ldots, r \\ x_\gamma = 0, \ \gamma = N \setminus \sigma, \quad x_\gamma \in R^{n-r} \\ r = 1, 2, \ldots, n\end{gathered} \tag{2.2}$$

has no non-trivial solutions in K for any $\lambda < 0 \ (\lambda > 0)$. Here $\alpha_{i_k 0}$ are basis elements for the cone.

The following example will illustrate the possibilities for third order forms. Considering the region

$$G = K\{\alpha_{10}, \ldots, \alpha_{n0}\} \times I, \quad I = [0, \infty[$$

let us determine whether the following model system is monotonically stable:

$$x_k^{\cdot} = -x_k V_k(x), \quad V_k(x) = \sum_{i,j=1}^{n} a_{kij} x_i x_j \tag{2.3}$$
$$a_{kij} = \text{const}, \quad k = 1, 2, \ldots, n$$

Such systems were introduced /8/ as model systems for stability analysis in the framework of a neutral linear approximation. The system satisfies the conditions of Krasnosel'skii's Lemma in the region of interest. Consequently, if the initial data lie in K, the trajectories of the system will not leave the cone. System (2.3) will be monotonically stable in K if, for any initial data in that region,

$$s(t) = \sum_{i=1}^{n} \alpha_{i0} x_i(t)$$

is a strictly monotone decreasing function of time along the trajectories of the system.

Thus, system (2.3) will be monotonically stable in the cone K if and only if the form

$$W(x) = \sum_{k=1}^{n} \alpha_{k0} x_k V_k(x)$$

is positive definite there. Hence it is obvious that the monotone stability properties of the system depend on the relationships among the basis elements $\{\alpha_{i0}\}$ of K and the coefficients a_{kij}.

Similar results may be obtained in a monotone stability analysis for the following system of differential equations, where the right-hand side is a special kind of polynomial of degree m:

$$x_{i_1}^{\cdot} = -x_{i_1} \sum_{i_2=i_1}^{n} \cdots \sum_{i_m=i_{m-1}}^{n} A_{i_1 \ldots i_m} x_{i_2} \cdots x_{i_m}$$
$$A_{i_1 \ldots i_m} = \mathrm{const}, \quad i_1 = 1, 2, \ldots, n$$

3. Coefficient criterion for a third-order form to be positive (negative) definite in a cone coinciding with a coordinate angle. To fix our ideas, let us consider the first quadrant. In the region $\{0 \leqslant \| x \| = \sqrt{(x, x)} < \infty,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}$, consider the form

$$W(x_1, x_2) = a_{30} x_1^3 + a_{21} x_2 x_1^2 + a_{12} x_1 x_2^2 + a_{03} x_2^3 \tag{3.1}$$

Obviously, this form can be written as

$$W(x_1, x_2) = x_1 (a_{30} x_1^2 + {}^2/_3 a_{21} x_1 x_2 + {}^1/_3 a_{12} x_2^2) + x_2 ({}^2/_3 a_{21} x_1^2 + {}^1/_3 a_{12} x_1 x_2 + a_{03} x_2^2)$$

Using our previously established criterion for a third-order form to be positive (negative) definite, we obtain the following result. The form (3.1) is negative definite in the first quadrant if and only if the systems of algebraic equations

$$\begin{cases} a_{30} x_1^2 + {}^2/_3 a_{21} x_1 x_2 + {}^1/_3 a_{12} x_2^2 = \lambda \\ {}^2/_3 a_{21} x_1^2 + {}^1/_3 a_{12} x_1 x_2 + a_{03} x_2^2 = \lambda \end{cases} \tag{3.2}$$
$$\begin{cases} x_1 = 0 \\ a_{03} x_2^3 \geqslant 0 \end{cases}, \quad \begin{cases} x_2 = 0 \\ a_{30} x_1^3 \geqslant 0 \end{cases}$$

considered in the above region, have no non-trivial solution in the first quadrant for any $\lambda > 0$. To determine whether these systems are solvable, proceed as follows. Evaluate the resultant of the first system and equate it to zero, on the assumption that a solution exists. Clearly, a_{03} and a_{30} must be negative. We thus obtain a biquadratic equation with parameter,

$$f(y^2) \equiv A_1 y^4 + \lambda A_2 y^2 + \lambda^2 A_3 = 0$$

Now use Newton's method to find an upper bound for the positive roots $z = y^2$. The first system in (3.2) is also unsolvable if the equation $f(z) = 0$ has complex roots, i.e., if

$$A_2^2 - 4A_1 A_3 < 0$$

Obviously, all the above algebraic arguments can be formalized in algorithmic form. We thus have a numerical method to test the form W for positive (negative) definiteness in a cone of R^2

4. Modification of Grujic's exponential stability theorem for large-scale systems. In the region

$$H = \{t \geqslant 0;\ 0 \leqslant \|x\| = \sqrt{(x, x)} < \infty,\ x \in R^n\}$$

consider a system of equations

$$x^{\cdot} = f(t, x),\ f(t, 0) \equiv 0,\ x \in R^n; \qquad f\colon I \times R^n \to R^n,\quad I = [0, \infty[\tag{4.1}$$

which splits into k interacting subsystems

$$\begin{gathered} x_s^{\cdot} = g_s(t, x_s) + h_s(t, x^\circ), \quad x_s \in R^{n_s}, \quad s = 1, 2, \ldots, k \\ g_s\colon I \times R^{n_s} \to R^{n_s}, \quad n_1 + \ldots + n_k = n \\ x^\circ = (x_1, \ldots, x_{s-1}, 0, x_{s+1}, \ldots, x_k) \\ h_s\colon I \times R^n \to R^{n_s} \end{gathered}$$

whose isolated subsystems have the form

$$x_s^{\cdot} = g_s(t, x_s),\ g_s(t, 0) \equiv 0,\ s = 1, 2, \ldots, k \tag{4.2}$$

Assume that the right-hand sides of system (4.1), and systems (4.2) are such as to guarantee the existence and uniqueness of solutions for any initial data in the region of interest.

The trivial solution of the compound system (4.1) will be analysed for stability by means of the Lyapunov function

$$V(t, x) = \sum_{s=1}^{k} V_s(t, x_s)$$

where $V_s(t, x_s)$ is a Lyapunov function for the s-th isolated subsystem of (4.1). Assume that all the solutions of subsystems (4.2) are either exponentially stable or exponentially unstable. Then there is a Lyapunov function $V_s(t, x_s)$ for each subsystem in (4.2), such that

$$\begin{gathered} c_{s1}\|x_s\|^m \leqslant V_s(t, x_s) \leqslant c_{s2}\|x_s\|^m \\ \mu_s c_{s3}\|x_s\|^m \leqslant V_s^{\cdot}(t, x_s) \leqslant \mu_s c_{s4}\|x_s\|^m \end{gathered} \tag{4.3}$$

where $c_{sl} > 0$ $(s = 1, 2, \ldots, k;\ l = 1, \ldots, 4)$ are real constants and μ_s takes values of -1 or +1, depending on whether the trivial solution of the s-th subsystem is exponentially stable or exponentially unstable.

We shall say that the interaction vector

$$h(t, x) = \operatorname{col}(h_1^T, \ldots, h_k^T)$$

belongs to class H_* if for any t and x

$$\sum_{s=1}^{k} (\nabla V_s, h_s) \leqslant \sum_{i_1=1}^{k} \cdots \sum_{i_m=i_{m-1}}^{k} \alpha_{i_1 \ldots i_m} \|x_{i_1}\| \cdots \|x_{i_m}\|$$

where $\alpha_{i_1 \ldots i_m}$ are real constants. The elements

$$\beta_{i_1 \ldots i_m} = \mu_{i_1} c_{i_1 4} \delta_{i_1 \ldots i_m} + \alpha_{i_1 \ldots i_m}$$

generate an m-th order form

$$W(y) = \sum_{i_1=1}^{k} \cdots \sum_{i_m=i_{m-1}}^{k} \beta_{i_1 \ldots i_m} y_{i_1} \cdots y_{i_m}$$

where $\delta_{i_1 \ldots i_m} = 1$, if $i_1 = i_2 = \ldots = i_m$ and $\delta_{i_1 \ldots i_m} = 0$ otherwise.

Theorem 2. Assume that for each s, subsystem (4.2) of system (4.1), considered in H, has a function $V_s(t, x_s)$ satisfying inequalities (4.3); assume moreover that the interaction matrix $h(t, x)$ belongs to class H_*. Then, if the m-th order form $W(y)$ is negative definite in the cone $(y \geqslant 0)$, the trivial solution of the compound system (4.1) is asymptotically stable in the large, uniformly in t_0 and x_0.

Grujic's well-known exponential stability theorem for large-scale systems of differential equations /1/ is based on the use of Lyapunov functions for the isolated subsystems, in the form of expressions involving quadratic estimates, and on the application of Sylvester's criterion.

Theorem 2 can be used, for example, to obtain more precise boundaries of the region of admissible parameter values for the equations of translational motion of an aircraft /9/, in which the equations are absolutely stable.

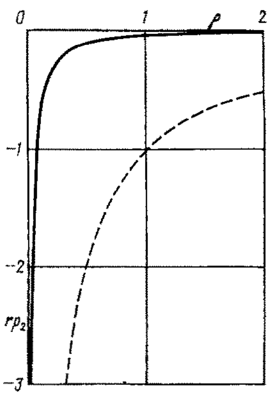
5. Investigation of the equations of translational motion of an aircraft using third-order forms. Consider the system of equations

$$x_s^{\cdot} = -\rho_s x_s + \sigma, \quad \sigma^{\cdot} = \sum \beta_s x_s + rp_2\sigma - f(\sigma) \tag{5.1}$$
$$\rho_s > 0, \; r > 0, \; p_2 < 0$$
$$0 < \sigma \cdot f(\sigma), \quad \sigma \neq 0; \quad f(0) = 0$$

which describes the translational motion of an aircraft. Throughout, $m = 1, 2, 3, 4$, and the summation is over s from $s = 1$ to $s = 4$.

Decoupling of Eqs.(5.1) yields two isolated subsystems:

$$x_s^{\cdot} = -\rho_s x_s \tag{5.2}$$
$$\sigma^{\cdot} = rp_2\sigma - f(\sigma) \tag{5.3}$$



Let us assume that system (5.1) has a Lyapunov function $V = c_1V_1 + c_2V_2$, where $V_1 = \| x \|^3$ and $V_2 = | \sigma |^3$ are Lyapunov functions for the isolated subsystems (5.2) and (5.3), respectively. The functions V_1 and V_2 satisfy the estimates

$$V_1^{\cdot} \leqslant -3\rho \| x \|^3, \quad V_2^{\cdot} < 3rp_2 | \sigma |^3$$

i.e., the isolated subsystems have exponentially stable trivial solutions. The total derivative along trajectories of system (5.1) satisfies the estimate

$$V^{\cdot} \leqslant -3c_1\rho \| x \|^3 + 3c_1 \| x \|^2 | \sigma | + 12c_2\beta \| x \| \times$$
$$| \sigma |^2 + 3c_2rp_2 | \sigma |^3 = W(y);$$
$$\beta = \max | \beta_s |, \quad y = \operatorname{col}(\| x \|, | \sigma |)$$

The form $W(y)$ is negative definite in the positive orthant $(y \geqslant 0)$ if and only if the system of non-linear equations

$$-3\rho c_1 y_1^2 + 2c_1y_1y_2 + 4c_2\beta y_2^2 = \lambda$$
$$c_1y_1^2 + 8c_2\beta y_1y_2 + 3c_2rp_2y_2^2 = \lambda$$

has no solutions in the cone $(y \geqslant 0)$ of R^2 for any $\lambda > 0$. Using the coefficient criteria for definiteness of a third-order form in two variables in the first orthant (see Sect.3), we obtain the following estimates for the region of admissible values:

$$81c_1\rho^2(rp_2)^2 - 72c_1c_2\beta\rho rp_2 - 768c_2^2\beta^3\rho - 1728c_2^2\beta^3\rho^2rp_2 + 6c_1rp_2 + 72c_2\beta\rho rp_2 > 0 \tag{5.4}$$
$$4c_1c_2\beta - 9c_1c_2\rho rp_2 + 12c_1c_2\rho\beta + 96c_2^2\beta^2\rho - 27c_1c_2\rho^2rp_2 + 288c_2^2\beta^2\rho^2 - c_1 > 0)$$

Stability analyses of system (5.1) have been undertaken by many authors (/9-12/ etc.), using a variety of methods and obtaining various results. In recent years similar automatic control systems have been investigated using scalar functions obtained by combining the components of a suitable vector-valued Lyapunov function. Thus, the following estimates have been established /9/:

$$\frac{(\max_s | c_1 + c_2\beta_s |)^2}{c_1 + c_2\rho r | p_2 |} < 1, \quad \frac{c_1 + c_2\beta}{c_1c_2\rho r | p_2 |} < 1$$

Our estimates (5.4) determine regions of admissible values for the parameters of system (5.1), in which it is absolutely stable in the large (the solid curve in the figure, where $\beta_s = 0$, $c_1 = c_2 = 1$), which are new compared with the previously determined regions /9/ (the dashed curve).

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ON THE STATIONARY MOTIONS IN A NEWTONIAN FIELD OF FORCE OF A BODY THAT ADMITS OF REGULAR POLYHEDRON SYMMETRY GROUPS*

R.S. SULIKASHVILI

The author's results /1-3/ on the stationary motions in a central Newtonian field of force when the centre of mass is assumed fixed, of a fixed system of particles of equal mass, located at the vertices of a regular polyhedron, are written in a general mathematical form and are extended to any fixed system whose mass distribution admits of one of the symmetry groups of a regular polyhedron (tetrahedron, octahedron, or icosahedron). It is shown that the results obtained earlier by considering the first terms of the Taylor expansion of the force function are preserved when account is taken of the full expression for the force function (potential). The stability of these solutions is investigated.

1. We consider the motion of a rigid body with a fixed point in a central Newtonian field of force. Let the fixed point G be at the centre of mass, and let the mass distribution in the body be invariant under transformations that belong to one of the discrete symmetry groups: the tetrahedron, octahedron, or icosahedron.

Let $O\xi\eta\zeta$ be a fixed coordinate system with origin at the attracting centre O, and $Gxyz$ a dimensionless coordinate system rigidly connected with the body (the scale of length is a characteristic dimension l of the body). The force function of Newtonian gravitation

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